

A POSSIBLE $SL_q(2)$ SUBSTRUCTURE OF THE STANDARD MODEL

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ABSTRACT: We examine a quantum group extension of the standard model with the symmetry $SU(3) \times SU(2) \times U(1) \times$ global $SL_q(2)$. The quantum fields of this extended model lie in the state space of the $SL_q(2)$ algebra. The normal modes or field quanta carry the factors $D_{mm'}^j(q|abcd)$, which are irreducible representations of $SL_q(2)$ (which is also the knot algebra). We describe these field quanta as quantum knots and set $(j, m, m') = 1/2(N, w, \pm r + 1)$ where the (N, w, r) are restricted to be (the number of crossings, the writhe, the rotation) respectively, of a classical knot.

There is an empirical one-to-one correspondence between the four quantum trefoils and the four families of elementary fermions, a correspondence that may be expressed as $(j, m, m') = 3(t, -t_3, -t_0)$, where the four quantum trefoils are labelled by (j, m, m') and where the four families are labelled in the standard model by the isotopic and hypercharge indices $(t, t_3, -t_0)$. We propose extending this correlation to all representations by attaching $D_{-3t-3t_0}^{3t}(q|abcd)$ to the field operator of every particle labelled by (t, t_3, t_0) in the standard model. Then the elementary fermions ($t = 1/2$) belong to the $j = 3/2$ representation of $SL_q(2)$. The elements of the fundamental representation $j = 1/2$ will be called preons and $D_{-3t-3t_0}^{3t}$ may be interpreted as describing the creation operator of a composite particle composed of elementary preons. $D_{mm'}^j$ also may be interpreted to describe a quantum knot when expressed as $D_{\frac{w}{2} \pm r + 1}^{N/2}$. These complementary descriptions may be understood as describing a composite particle of N preons bound by a knotted boson field with N crossings.

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1 Introduction

The general notion that the elementary particles are topologically stabilized has had a long history beginning with Kelvin.¹ In recent times a classical knot related to the Skyrme soliton has been described by Faddeev and Niemi.²

There is also an apparently unrelated preon literature beginning with the work of Pati and Salam.³ It turns out, however, that the knot and preon conjectures are not unrelated but may be formulated as complementary expressions of $SLq(2)$ symmetry.

To review the knot conjecture first note that the familiar knots of magnetic fields are macroscopic manifestations of the electroweak field. It is then natural to consider knots of electroweak field that are microscopic and quantized as well. Since these would be observed as solitonic in virtue of both their topological and quantum stability, it is also natural to ask if the known elementary particles might also be quantized knots of field. If they are, one expects that the most elementary particles, namely the elementary fermions, are also the most elementary quantum knots, namely the quantum trefoils. This possibility is suggested by the fact that there are 4 quantum trefoils and 4 classes of elementary fermions, and is supported by a unique one-to-one correspondence between the topological characterization of the 4 quantum trefoils and the quantum numbers of the 4 fermionic classes. To define a quantum knot we shall first record the irreducible representations of the knot algebra ($SLq(2)$).

2 Irreducible Representations of the Knot Algebra⁴

The $2j + 1$ dimensional representation of $SLq(2)$ may be written as follows:

$$D_{mm'}^j(a, b, c, d) = \sum_{s < n_+, t < n_-} \mathcal{A}_{mm'}^j(q, s, t) \delta(s + t, n'_+) a^s b^{(n_+ - s)} c^t d^{(n_- - t)} \quad (2.1)$$

where

$$n_{\pm} = j \pm m \quad (2.2)$$

$$n'_{\pm} = j \pm m' \quad (2.3)$$

and the arguments (a, b, c, d) satisfy the knot algebra:⁵

$$\begin{aligned}
ab &= qba & bd &= qdb & bc &= cb & ad - qbc &= 1 \\
ac &= qca & cd &= qdc & da - q_1cb &= 1
\end{aligned} \tag{A}$$

where $q_1 = q^{-1}$.

The $\mathcal{A}_{mm'}^j$ are q -deformations of the Wigner coefficients that appear in irreducible representations of $SU(2)$.

The knot algebra (A) and hence $D_{mm'}^j(a, b, c, d)$ are defined only up to the gauge transformation

$$\begin{aligned}
U_a(1) : \quad & a' = e^{i\varphi_a} a \\
& d' = e^{-i\varphi_a} d \\
U_b(1) : \quad & b' = e^{i\varphi_b} b \\
& c' = e^{-i\varphi_b} c
\end{aligned} \tag{2.4}$$

Eqns (2.4) leave the algebra (A) invariant and induce on the elements of every representation the following $U_a(1) \times U_b(1)$ gauge transformation⁶

$$D_{mm'}^j(a', b', c', d') = e^{i(\varphi_a + \varphi_b)m} e^{i(\varphi_a - \varphi_b)m'} D_{mm'}^j(a, b, c, d) \tag{2.5}$$

3 Quantization of the Knot

Following the example of the quantization of angular momentum by representations of $D_{mm'}^j$ of $SU(2)$ where the indices (j, m, m') refer to components of the angular momentum, we shall quantize the kinematics of the knot with representations of the knot algebra where the indices on $D_{mm'}^j$ are now related to (N, w, r) ⁷, the number of crossings, the writhe, and the rotation of the corresponding classical knot by

$$\begin{aligned}
j &= \frac{N}{2} \\
m &= \frac{w}{2} \\
m' &= \frac{\pm r + 1}{2}
\end{aligned} \tag{3.1}$$

The relations (3.1) satisfy the following restrictions:

(a) Of the set (N, w, r) only N is never negative and therefore corresponds to j which is also never negative

(b) Half-integer representations require the factor $1/2$

(c) $(2m)$ and $(2m')$, belonging to the same representation, are of the same parity, while the knot constraints require w and r to be of opposite parity.

Since the spectra of (j, m, m') are restricted by $SLq(2)$, and the spectra of (N, w, r) are restricted by knot topology, the states of the quantized knot are thus jointly restricted by both $SLq(2)$ and the knot topology. The equations (3.1) then establish a correspondence between a quantized knot described by $D_{\frac{w \pm r + 1}{2}}^{N/2}$ and a classical knot described by (N, w, r) , but the correspondence is not one-to-one.

Let

$$\Psi_{\frac{w \pm r + 1}{2}}^{N/2} = D_{\frac{w \pm r + 1}{2}}^{N/2} \sum_n c_n |n\rangle \quad (3.2)$$

where $\sum c_n |n\rangle$ lies in the state space defined by the knot algebra (A). The states, $|n\rangle$, forming a basis in this space are eigenstates of the commuting elements, b and c , with eigenvalues $\sim q^{n(6)}$.

Here $\Psi_{\frac{w \pm r + 1}{2}}^{N/2}$ is intended to describe a generic quantum knot for which the Hamiltonian has not been specified. In (3.2) $D_{\frac{w \pm r + 1}{2}}^{N/2}$ is a kinematic factor that resembles the spherical harmonic factor in $Y_m^\ell(\theta, \varphi)R(r)$, an eigenstate of a spherically symmetric Hamiltonian.

There are, as usual, important differences between the quantum construction and its classical image. In particular there are only two classical trefoils: $(w, r) = (3, 2)$ and $(-3, 2)$ while there are four quantum trefoils labelled by

$$(j, m, m') = \left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}\right), \left(\frac{3}{2}, -\frac{3}{2}, \frac{3}{2}\right), \left(\frac{3}{2}, \frac{3}{2}, -\frac{1}{2}\right), \left(\frac{3}{2}, -\frac{3}{2}, -\frac{1}{2}\right)$$

where $m' = -\frac{1}{2}$ corresponds to $m' = \frac{-r+1}{2}$ by (3.1) with $r = 2$.

The classical trefoils $(w, 2)$ and $(w, -2)$ are topologically not distinguishable. The corresponding quantum trefoils $(w, 2)$ and $(w, -2)$, when realized as elementary fermions, are distinguished by different values of the hypercharge, as we shall see. In the following, when (w, r) refers to a quantum trefoil, r may have either sign, and we shall write simply $m' = \frac{r+1}{2}$.

There are also quantum states for which $j < \frac{3}{2}$. By (3.1) these correspond to a classical image for which $N < 3$ and do not qualify as classical knots but may be described as twisted loops. We shall see that these $j < \frac{3}{2}$ states may be realized as preons.

4 Field Theory and Charges of Quantum Trefoils

One may construct a field theory of the quantum trefoils $D_{mm'}^{3/2}$ by attaching $D_{mm'}^{3/2}$ to a standard fermion field operator $\psi(x)$ as follows:

$$\Psi_{mm'}^{3/2} = \psi(x) D_{mm'}^{3/2} \quad (4.1)$$

By (2.5), the field operator $\Psi_{mm'}^{3/2}$ also transforms under the gauge transformations $U_a(1) \times U_b(1)$. One must now require that the new field action be invariant under $U_a(1) \times U_b(1)$ since the relabelling of the algebra described by (2.4) cannot affect the physics. Then by Noether's theorem there will be one conserved charge associated with $U_a(1)$ and a second conserved charge associated with $U_b(1)$. Then by (2.5) and (3.1) these charges may be defined by

$$Q(w) \equiv -k_w m = -k_w \frac{w}{2} \quad w = \pm 3 \quad (4.2)$$

$$Q(r) \equiv -k_r m' = -k_r \frac{r+1}{2} \quad r = \pm 2 \quad (4.3)$$

and may be referred to as the writhe and rotation charges. Here k_w and k_r are undetermined constants with the dimensions of an electric charge. In terms of $Q(w)$ and $Q(r)$, the $U_a(1) \times U_b(1)$ transformations on $\Psi_{mm'}^{3/2}$ become

$$\Psi_{mm'}^{3/2'} = e^{\frac{-i}{k_w} Q(w) \varphi(w)} e^{\frac{-i}{k_r} Q(r) \varphi(r)} \Psi_{mm'}^{3/2} \quad (4.4)$$

where $\varphi(w) = \varphi_a + \varphi_b$ and $\varphi(r) = \varphi_a - \varphi_b$ by (2.5). We next make a direct comparison between the $Q(w)$ and $Q(r)$ charges of the quantum trefoil and the charge and hypercharge of the four fermion families, each denoted by (f_1, f_2, f_3) in Table 4.1.⁴

Standard Representation					Trefoil Representation				
(f_1, f_2, f_3)	\underline{t}	\underline{t}_3	\underline{t}_0	\underline{Q}_e	(w, r)	$D_{\frac{w}{2} \frac{r+1}{2}}^{N/2}$	\underline{Q}_w	\underline{Q}_r	$\underline{Q}_w + \underline{Q}_r$
(e, μ, τ)	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-e$	(3,2)	$D_{\frac{3}{2} \frac{3}{2}}^{3/2}$	$-k \left(\frac{3}{2}\right)$	$-k \left(\frac{3}{2}\right)$	$-3k$
$(\nu_e, \nu_\mu, \nu_\tau)$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	(-3,2)	$D_{-\frac{3}{2} \frac{3}{2}}^{3/2}$	$-k \left(-\frac{3}{2}\right)$	$-k \left(\frac{3}{2}\right)$	0
(d, s, b)	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{3}e$	(3,-2)	$D_{\frac{3}{2} -\frac{1}{2}}^{3/2}$	$-k \left(\frac{3}{2}\right)$	$-k \left(-\frac{1}{2}\right)$	$-k$
(u, c, t)	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{2}{3}e$	(-3,-2)	$D_{-\frac{3}{2} -\frac{1}{2}}^{3/2}$	$-k \left(-\frac{3}{2}\right)$	$-k \left(-\frac{1}{2}\right)$	$2k$

Table 4.1

In Table (4.1) we have assumed a single value of k ;

$$k_r = k_w = k \quad (4.5)$$

which is also the same for all trefoils. If we set $k = e/3$, we find that the four fermion families are related to the four quantum trefoils as follows:

$$Q_w = et_3 \quad (4.6)$$

$$Q_r = et_0 \quad (4.7)$$

$$Q_w + Q_r = Q_e \quad (4.8)$$

in agreement with the standard model where there is the independent relation for the electric charge

$$Q_e = e(t_3 + t_0) \quad (4.9)$$

If one aligns the trefoils and the fermion families in some order different from that in Table 4.1, one needs more than a single value of k . It is important that we choose $k_r = k_w$ and that we also choose a single value of k for the four quantum trefoils. Note that it is also not possible to exchange t_3 and t_0 in (4.6) and (4.7). Therefore the correspondence between the four fermion families and the four trefoils, as well as the value of k as $e/3$, are

empirically fixed and unique. This complete correspondence justifies the representation of each of the four fermion families by a quantum trefoil.

The correspondence between the quantum trefoils and the elementary fermions may also be summarized by the following relations which may also be read directly from Table (4.1).

$$t = \frac{N}{6} \quad t = \frac{j}{3} \quad (4.10)$$

$$t_3 = -\frac{w}{6} \quad t_3 = -\frac{m}{3} \quad (4.11)$$

$$t_0 = -\frac{r+1}{6} \quad t_0 = -\frac{m'}{3} \quad (4.12)$$

$$Q_e = -\frac{e}{6}(w+r+1) \quad Q_e = -\frac{e}{3}(m+m') \quad (4.13)$$

Note also that

$$Q_e = -\frac{e}{N} \left(\frac{w+r+1}{2} \right) \quad Q_e = -e \frac{(m+m')}{2j} \quad (4.14)$$

holds for all the elementary fermions.

While Q, t_3 and t_0 are defined in the standard model with respect to $SU(2) \times U(1)$, here Q, t_3 and t_0 are defined with respect to the gauge transformations $U_a(1) \times U_b(1)$ of the knot algebra. They are also described by (w, r) and (m, m') as shown in Table (4.1) and equations (4.11) - (4.13). In the limit of the standard model t_3 and t_0 assume their usual meaning in the $SU(2)$ and $U(1)$ representations.

The kinematic factors $D_{m,m'}^j$ labelled by quantum numbers (j, m, m') or (t, t_3, t_0) are multinomials lying in the knot algebra (A) and are explicitly given by (2.1). These multinomials are associated with the knot (N, w, r) and, like the Jones polynomial, label the knot.

We incorporate equations (4.10-4.12) into (4.1) as follows

$$\Psi^{3/2}(t_3, t_0, n) = \psi(t_3, t_0, n) D_{-3t_3-3t_0}^{3/2} |n\rangle \quad (4.15)$$

where $\psi(t_3, t_0, n)$ is the quantum field of the standard model that represents the fermion with electroweak $SU(2) \times U(1)$ quantum numbers (t_3, t_0) . Here $|n\rangle$ lies in the state space defined by the knot algebra where $n = 0, 1, 2$ labels the generation, e.g. (e, μ, τ) . Then

$D_{-3t_3-3t_0}^{3/2} |n\rangle$ may be regarded as an “internal state function” reminiscent of a classical knot and providing substructure to the elementary quantum fields of the standard model.

We shall now propose that the non-trivial correspondence embodied in Table (4.1) and expressed by (4.15) for the elementary fermions holds more generally in the following form

$$\Psi_{t_3 t_0}^t(n) = \psi(t, t_3, t_0, n) D_{-3t_3-3t_0}^{3t} |n\rangle \quad (4.16)$$

i.e., we assume that (t, t_3, t_0) are related to (j, m, m') just as in the special case $t = \frac{1}{2}$:

$$\begin{aligned} 3t &= j \\ 3t_3 &= -m \\ 3t_0 &= -m' \end{aligned} \quad (4.17)$$

In other words we assume that there is an underlying $SLq(2)$ symmetry of the elementary particles that may be expressed through the internal state functions $D_{mm'}^j |n\rangle$. For $j \geq 1$ not all states (m, m') of $D_{mm'}^j$ are filled. The occupied states are labelled by $D_{-3t_3-3t_0}^{3t}$ according to (4.17) and are determined by the intersection of the electroweak $SU(2) \times U(1)$ and the $SUq(2)$ symmetries. The $|n\rangle$ are intended to represent the possible states of excitation of the quantum knot. For example, the analogue of Table 4.1 for the elementary fermions is Table 4.2 for the elementary bosons of the Weinberg-Salam model

	\underline{t}	$\underline{t_3}$	$\underline{t_0}$	$\underline{D_{-3t_3-3t_0}^{3t}}$
W^+	1	1	0	D_{-30}^3
W^-	1	-1	0	D_{30}^3
W^3	1	0	0	D_{00}^3
W^0	0	0	0	D_{00}^0

Table 4.2

We adopt the following rule:

If a particle is labelled in the standard model by electroweak quantum numbers (t, t_3, t_0) then attach to the quantum field operator of that particle the factor $D_{-3t_3-3t_0}^{3t}(a, b, c, d)$.

This factor is to be understood as an element of the $j = 3t$ representation of the $SLq(2)$ algebra and may be interpreted as the replacement of the point particle of the standard model by a solitonic structure described solely by this factor.

5 The Electroweak Interactions⁸

In the $SLq(2)$ model the solitonic fermions interact by the emission and absorption of solitonic bosons. Denote the generic fermion-boson interaction by

$$\bar{\mathcal{F}}'' \mathcal{B}' \mathcal{F} \quad (5.1)$$

where

$$\mathcal{F} = \mathbf{F}(p, s, t_3, t_0) \left(D_{-3t_3-3t_0}^{3/2} \right) |n > \quad (5.2)$$

$$\bar{\mathcal{F}}'' = < n'' | \left(\bar{D}_{-3t_3-3t_0}^{3/2} \right)'' \bar{\mathbf{F}}''(p, s, t_3, t_0) \quad (5.3)$$

$$\mathcal{B}' = \mathbf{B}'(p, s, t_3, t_0) \left(D_{-3t_3-3t_0}^3 \right)' \quad (5.4)$$

and the pair (p, s) refer to momentum and spin.

Then (5.1) becomes

$$(\bar{\mathbf{F}}'' \mathbf{B}' \mathbf{F}) < n'' | \bar{D}_{-3t_3'-3t_0'}^{3/2} D_{-3t_3'-3t_0'}^3 D_{-3t_3-3t_0}^{3/2} | n > \quad (5.5)$$

The matrix elements of the standard model will then be modified by the following form factors.

$$< n'' | \bar{D}_{-3t_3'-3t_0'}^{3/2} D_{-3t_3'-3t_0'}^3 D_{-3t_3-3t_0}^{3/2} | n > \quad (5.6)$$

Here n and n'' take on the values 0, 1, 2 corresponding to the 3 generations in each family of fermions. These form factors are 2 parameter numerical functions that are in principle observable.⁸ To calculate them one needs the solitonic factors $D_{mm'}^j(a, b, c, d)$ shown in Tables (5.1) and (5.2).

(f_1, f_2, f_3)	\underline{t}	$\underline{t_3}$	$\underline{t_0}$	\underline{Q}	$\underline{D_{-3t_3-3t_0}^{3t}}$
(e, μ, τ)	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-e$	$D_{\frac{3}{2}\frac{3}{2}}^{3/2} \sim a^3$
$(\nu_e, \nu_\mu, \nu_\tau)$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	$D_{-\frac{3}{2}\frac{3}{2}}^{3/2} \sim c^3$
(d, s, b)	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{3}e$	$D_{\frac{3}{2}\frac{1}{2}}^{3/2} \sim ab^2$
(u, c, t)	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{2}{3}e$	$D_{-\frac{3}{2}\frac{1}{2}}^{3/2} \sim cd^2$

Table 5.1

	\underline{t}	$\underline{t_3}$	$\underline{t_0}$	\underline{Q}	$\underline{D_{-3t_3-3t_0}^{3t}}$
W^+	1	1	0	e	$D_{-30}^3 \sim c^3 d^3$
W^-	1	-1	0	$-e$	$D_{30}^3 \sim a^3 b^3$
W^3	1	0	0	0	$D_{00}^3 \sim f_3(b, c)$
W^0	0	0	0	0	$D_{00}^0 \sim f_0(b, c)$

Table 5.2

The solitonic factors are computed according to (2.1) and are all monomials except for the neutral W^0 and W^3 . The numerical factors $\mathcal{A}_{mm'}^3$ have been dropped but may be computed according to

$$\mathcal{A}_{mm'}^j = \left[\frac{\langle n'_+ \rangle_{q_1}! \langle n'_- \rangle_{q_1}!}{\langle n_+ \rangle_{q_1}! \langle n_- \rangle_{q_1}!} \right]^{\frac{1}{2}} \langle \begin{smallmatrix} n_+ \\ s \end{smallmatrix} \rangle_{q_1} \langle \begin{smallmatrix} n_- \\ t \end{smallmatrix} \rangle_{q_1} \quad (5.7)$$

where

$$\langle \begin{smallmatrix} n \\ s \end{smallmatrix} \rangle_q = \frac{\langle n \rangle_q!}{\langle n-s \rangle_q! \langle s \rangle_q!} \quad \text{with} \quad \langle n \rangle_q = \frac{q^n - 1}{q - 1}; \quad q_1 = q^{-1} \quad (5.8)$$

6 The Preon Representations.⁴

The elementary fermions already discussed are found in the $j = 3/2$ representation while the electroweak bosons lie in the $j = 3$ representation. We shall now consider the adjoint

($j = 1$) and fundamental ($j = 1/2$) representations. These are shown in Tables (6.1) and (6.2) again calculated with (2.1) but ignoring the numerical coefficients.

$$D^{1/2} :$$

$m \backslash m'$	$\frac{1}{2}$	$-\frac{1}{2}$
$\frac{1}{2}$	a	b
$-\frac{1}{2}$	c	d

Table 6.1

$$D^1 :$$

$m \backslash m'$	1	0	-1
1	a^2	ab	b^2
0	ac	$ad + bc$	bd
-1	c^2	cd	d^2

Table 6.2

We shall refer to the members of the $D^{1/2}$ and D^1 representations as preons and bosonic preons respectively.

To determine (t_3, t_0, Q) for the preons and bosonic preons we shall extend the relations (4.11), (4.12), (4.13), empirically established for the elementary fermions, then extended to the electroweak bosons and generally embodied in $D_{-3t_3-3t_0}^{3t}$. The results for preons and bosonic preons are shown in Tables (6.3) and (6.4).

Fermionic Preons $t = 1/6$

	t	t_3	t_0	Q
a	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{e}{3}$
b	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	0
d	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{e}{3}$
c	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	0

Table 6.3

Bosonic Preons $t = 1/3$

	$\underline{t_3}$	$\underline{t_0}$	$\underline{Q/e}$	$\underline{D_{mm'}^1}$		$\underline{t_3}$	$\underline{t_0}$	$\underline{Q/e}$	$\underline{D_{mm'}^1}$		$\underline{t_3}$	$\underline{t_0}$	$\underline{Q/e}$	$\underline{D_{mm'}^1}$
D_{11}^1	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{2}{3}$	a^2	D_{01}^1	0	$-\frac{1}{3}$	$-\frac{1}{3}$	ac	D_{-11}^1	$\frac{1}{3}$	$-\frac{1}{3}$	0	c^2
D_{10}^1	$-\frac{1}{3}$	0	$-\frac{1}{3}$	ab	D_{00}^1	0	0	0	$ad + bc$	D_{-10}^1	$\frac{1}{3}$	0	$\frac{1}{3}$	cd
D_{1-1}^1	$-\frac{1}{3}$	$\frac{1}{3}$	0	b^2	D_{0-1}^1	0	$\frac{1}{3}$	$\frac{1}{3}$	bd	D_{-1-1}^1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	d^2

Table 6.4

By Table (6.3) there are two preons, a and b , charged and neutral, respectively, and their respective antiparticles, d and c . Other particles may be regarded as built out of the preons (a, b, c, d) and the values of (t_3, t_0, Q) for all of these composite particles may be obtained by adding the (t_3, t_0, Q) of each of the constituent preons. Therefore the factors $D_{-3t_3-3t_0}^{3t}$ may be read in two ways: (a) as describing creation operators for quantum knots representing the internal state of a composite particle or (b) as a product of creation operators for the component preons.

The preceding remarks are illustrated in Tables (5.1), (5.2), (6.3) and (6.4) where $D_{-3t_3-3t_0}^{3t}$ and (Q, t, t_3, t_0) are summarized for preons, bosonic preons, elementary fermions, and weak bosons.

To show that the preon interpretation holds in all representations let us rewrite (2.1) by introducing (n_a, n_b, n_c, n_d) the exponents of (a, b, c, d) :

$$n_a = s \tag{6.1}$$

$$n_b = n_+ - s$$

$$n_c = t \tag{6.2}$$

$$n_d = n_- - t$$

Then

$$n_+ = n_a + n_b \tag{6.3}$$

$$n_- = n_c + n_d$$

$$\begin{aligned} n'_+ &= n_a + n_c \\ n'_- &= n_b + n_d \end{aligned} \tag{6.4}$$

and

$$n_a + n_b + n_c + n_d = n_+ + n_- = 2j (= N = 6t) \tag{6.5}$$

$$n_a + n_b - n_c - n_d = n_+ - n_- = 2m (= w = -6t_3) \tag{6.6}$$

$$n_a + n_c - n_b - n_d = n'_+ - n'_- = 2m' (= r + 1 = -6t_0) \tag{6.7}$$

In the preon interpretation of $D_{mm'}^j$ the (a, b, c, d) are regarded as creation operators for (a, b, c, d) particles and the (n_a, n_b, n_c, n_d) are the numbers of (a, b, c, d) preons in each term. These will vary from term to term but the left sides of Equations (6.5) - (6.7) remain the same in all terms contributing to $D_{mm'}^j$ and they also have simple meanings.

Equations (6.5) - (6.7) may be rewritten as

$$t = \frac{1}{6}(n_a + n_b + n_c + n_d) \tag{6.8}$$

$$t_3 = -\frac{1}{6}(n_a + n_b - n_c - n_d) \tag{6.9}$$

$$t_0 = -\frac{1}{6}(n_a - n_b + n_c - n_d) \tag{6.10}$$

By (4.9), (6.9) and (6.10)

$$Q = -\frac{e}{3}(n_a - n_d)$$

The representations already considered, (Tables (5.1), (5.2), and (6.3)) illustrate special cases of these general relations that express (t, t_3, t_0) of a composite particle in terms of the charges of the preonic constituents,

7 The Complementary Models.⁹

The equations (6.5), (6.6), (6.7), may also be read as knot relations

$$n_a + n_b + n_c + n_d = N \quad (7.1)$$

$$n_a + n_b - n_c - n_d = w \quad (7.2)$$

$$n_a + n_c - n_b - n_d = r + 1 \quad (7.3)$$

Equation (7.1) states that the total number of preons equals the number of crossings (N). Since we shall assume that the preons are fermions, the knot is a fermion or boson depending on whether the number of crossings is odd or even.

The meaning of (7.2) and (7.3) becomes clearer if we note that a and d are antiparticles since they have opposite charge and hypercharge, while b and c are neutral antiparticles with opposite values of the hypercharge. We may therefore introduce the "preon numbers":

$$\nu_a = n_a - n_d \quad (7.4)$$

$$\nu_b = n_b - n_c \quad (7.5)$$

Then (7.2) and (7.3) may be rewritten as

$$\nu_a + \nu_b = w \quad (7.6)$$

$$\nu_a - \nu_b = r + 1 \quad (7.7)$$

By (7.6) and (7.7) the conservation of writhe and rotation is equivalent to the conservation of the preon numbers.

These considerations have led us to the position that the symmetry of a solitonic elementary particle, that is described by representations of the $SLq(2)$ algebra, may be expressed in any of the following ways:

$$D_{mm'}^j = D_{-3t_3-3t_0}^{3t} = D_{\frac{w}{2} \frac{r+t}{2}}^{N/2} = \hat{D}_{\nu_a \nu_b}^{N'} \quad (7.8)$$

where N' is the total number of preons.

We interpret the different forms of $D_{mm'}^j$ as showing that different aspects of the solitonic particle all display the same $SLq(2)$ symmetry.

In terms of (N, w, r)

$$D^{\frac{N}{2} \frac{w}{2} \frac{r+1}{2}}(q|abcd) = \left[\frac{\langle n'_+ \rangle_{q_1}! \langle n'_- \rangle_{q_1}!}{\langle n_+ \rangle_{q_1}! \langle n_- \rangle_{q_1}!} \right]^{1/2} \sum_{\substack{0 \leq s \leq n_+ \\ 0 \leq t \leq n_-}} \left\langle \begin{matrix} n_+ \\ s \end{matrix} \right\rangle_{q_1} \left\langle \begin{matrix} n_- \\ t \end{matrix} \right\rangle_{q_1} \delta(s+t, n'_+) a^s b^{n_+-s} c^t d^{n_--t} \quad (7.9)$$

where

$$n_{\pm} = \frac{1}{2}[N \pm w] \quad (7.10)$$

$$n'_{\pm} = \frac{1}{2}[N \pm (r+1)] \quad (7.11)$$

The complementary description expressed in terms of the population numbers (n_a, n_b, n_c, n_d) is

$$D^j_{mm'} = \hat{D}^{N'}_{\nu_a \nu_b} \quad (7.12)$$

where

$$\hat{D}^{N'}_{\nu_a \nu_b} = \left[\frac{\langle n_a + n_c \rangle_{q_1}! \langle n_b + n_d \rangle_{q_1}!}{\langle n_a + n_b \rangle_{q_1}! \langle n_c + n_d \rangle_{q_1}!} \right]^{1/2} \sum_{\substack{N' \geq n_a, n_b \geq 0 \\ N' \geq n_c, n_d \geq 0}} \left\langle \begin{matrix} n_a + n_b \\ n_a \end{matrix} \right\rangle_{q_1} \left\langle \begin{matrix} n_c + n_d \\ n_c \end{matrix} \right\rangle_{q_1} a^{n_a} b^{n_b} c^{n_c} d^{n_d} \quad (7.13)$$

These complementary representations (7.9) and (7.13) are related by

$$\hat{D}^{N'}_{\nu_a \nu_b} = \sum_{Nwr} \delta(N', N) \delta(\nu_a + \nu_b, w) \delta(\nu_a - \nu_b, r+1) D^{\frac{N}{2} \frac{w}{2} \frac{r+1}{2}} \quad (7.14)$$

where N' is the number of preons and N is the number of crossings.

For the fundamental and adjoint representations we have $j = 1/2$ and $j = 1$ respectively and therefore $N = 1$ or $N = 2$, where N is the number of crossings. These do not describe knots but twisted loops. We may however still compute w and r in the same way as for knots. Although these twisted loops would not have the topological stability of knots, they could be prevented from unrolling by a dynamical stability of w and r or equivalently by the conservation of the preon numbers.

Viewed as a knot, a fermion becomes a boson when the number of crossings is changed by adding or subtracting a curl. This picture is consistent with the complementary view of a curl as an opened preon loop.

8 Gluon Charge.⁹

The previous considerations are based on electroweak physics. To describe the strong interactions it is necessary according to standard theory to introduce $SU(3)$ charge. We shall therefore assume that each of the four preon operators appears in triplicate (a_i, b_i, c_i, d_i) where $i = R, Y, G$, without changing the algebra (A) . These colored preon operators provide a basis for the fundamental representation of $SU(3)$ just as the colored quark operators do in standard theory. To adapt the electroweak operators to the requirements of gluon fields we make the following replacements:

$$\text{leptons: } a^3 \rightarrow \epsilon^{ijk} a_i a_j a_k \quad (8.1)$$

$$\text{neutrinos: } c^3 \rightarrow \epsilon^{ijk} c_i c_j c_k \quad (8.2)$$

$$\text{down quarks: } ab^2 \rightarrow a_i (\bar{b}^k b_k) \quad (8.3)$$

$$\text{up quarks: } cd^2 \rightarrow c_i (\bar{d}^k d_k) \quad (8.4)$$

where \bar{b}^k and $\bar{d}^k \sim \bar{3}$ representation of $SU(3)$. Here $(i, j, k) = (R, Y, G)$ and $(a_i b_i c_i d_i)$ are creation operators for colored preons. Then the leptons and neutrinos are color singlets while the quark states correspond to the fundamental representation of $SU(3)$, as required by standard theory. (Here b and \bar{b} , as well as d and \bar{d} , are antiparticles with respect to $SU(3)$ but have the same values of t_3 and t_0 .)

9 The Elementary Fermions as Preonic Trefoils⁹

Since the number of crossings equals the number of preons, one may speculate that there is one preon at each crossing if both preons and crossings are considered pointlike. If the pointlike crossings are labelled $(\vec{x}_1 \vec{x}_2 \vec{x}_3)$, then by (8.1)-(8.4) the wave functions of the trefoils representing leptons (ℓ), neutrinos (ν), down quarks (d), up quarks (u) are as follows:

$$\Psi_\ell(\vec{x}_1 \vec{x}_2 \vec{x}_3) = \epsilon^{ijk} \psi_i(a|\vec{x}_1) \psi_j(a|\vec{x}_2) \psi_k(a|\vec{x}_3) \quad (9.1)$$

$$\Psi_\nu(\vec{x}_1 \vec{x}_2 \vec{x}_3) = \epsilon^{ijk} \psi_i(c|\vec{x}_1) \psi_j(c|\vec{x}_2) \psi_k(c|\vec{x}_3) \quad (9.2)$$

$$\Psi_d(\vec{x}_1\vec{x}_2\vec{x}_3) = \psi_i(a|\vec{x}_1)\bar{\psi}^j(\bar{b}|\vec{x}_2)\psi_j(b|\vec{x}_3) \quad (9.3)$$

$$\Psi_u(\vec{x}_1\vec{x}_2\vec{x}_3) = \psi_i(c|\vec{x}_1)\bar{\psi}^j(\bar{d}|\vec{x}_2)\psi_j(d|\vec{x}_3) \quad (9.4)$$

where $i = (R, Y, G)$ and $\psi_i(a|\vec{x}) \dots \psi_i(d|\vec{x})$ are colored δ -like functions localizing the preons at the crossings.

Then the wave function of a lepton describes a *singlet* trefoil particle containing three preons of charge $(-e/3)$ and hypercharge $(-e/6)$. The corresponding characterization of a neutrino describes a *singlet* trefoil containing three neutral preons of hypercharge $(-e/6)$.

The wave function of a down quark describes a colored trefoil particle containing one a preon with charge $(-e/3)$ and hypercharge $(-e/6)$ and two neutral b preons with hypercharge $(e/6)$. The corresponding characterization of an up-quark describes a colored trefoil containing two charged d preons with charges $(e/3)$ and hypercharge $(e/6)$, and one neutral c preon with hypercharge $(-e/6)$.

This hypothetical structure is held together by the trefoil of fields connecting the charged preons. A search for this kind of substructure depends critically on the mass of the conjectured preons and the strength with which they are bound.

10 Preons as Physical Particles.⁴

We have so far viewed the preons mainly as a simple way to describe the algebraic structure of the knot polynomials. If these preons are in fact physical particles, the following decay modes of the quarks are possible.

$$\text{Down quarks: } \mathcal{D}_{\frac{3}{2}-\frac{1}{2}}^{3/2} \rightarrow \mathcal{D}_{\frac{1}{2}\frac{1}{2}}^{1/2} + \mathcal{D}_{1-1}^1, \quad (ab^2 \rightarrow a + b^2)$$

or

$$\text{Up quarks: } \mathcal{D}_{-\frac{3}{2}-\frac{1}{2}}^{3/2} \rightarrow \mathcal{D}_{-\frac{1}{2}\frac{1}{2}}^{1/2} + \mathcal{D}_{-1-1}^1, \quad (cd^2 \rightarrow c + d^2)$$

and the preons could play an intermediary role as virtual particles in quark processes.

The simple knot model predicts an unlimited number of excited states^{2,3} but it appears that there are only three generations, e.g. (d, s, b) . According to the preon scenario, however, it may be possible to avoid this problem by showing that the quarks will dissociate into

preons if given a critical “dissociation energy” less than that needed to reach the level of the fourth predicted flavor. In that case one would also expect the formation of a preon-quark plasma at sufficiently high temperatures.

It may be possible to study the thermodynamics of the plasma composed of quarks and these hypothetical particles.

Since the a and \bar{a} particles are charged ($\pm e/3$) one should expect their electro-production according to

$$e^+ + e^- \rightarrow a + \bar{a} \dots$$

at sufficiently high energies of a colliding (e^+, e^-) pair.

Since the preons are necessarily assumed to be pointlike they must also be very heavy. If the trefoil model is considered seriously for the leptons and neutrinos, then the binding energy must nearly compensate the mass of the very heavy constituent preons with a correspondingly higher melting temperature for the leptons and neutrinos.

11 Summary and Comments

In this paper the quantum knot has been characterized kinematically by $D_{mm'}^j(abcd)$, an element of an irreducible representation of the knot algebra $SLq(2)$ with

$$(j, m, m') = \frac{1}{2} (N, w, \pm r + 1) \quad (11.1)$$

where the spectrum of (j, m, m') is limited by $SLq(2)$ and the spectrum of (N, w, r) is restricted by the topology of a classical knot. The pair (w, r) are topological constants of the classical motion and the pair (m, m') are quantum constants of the motion by virtue of the $U_a(1) \times U_b(1)$ invariance of the $SLq(2)$ algebra.

When the 12 elementary fermions are described as 3 states of excitation of 4 quantum trefoils, each quantum trefoil corresponds to one family of 3 fermions. The correspondence is unique and is expressed by the empirical relation

$$(j, m, m') = 3(t, -t_3, -t_0) \quad (11.2)$$

where (j, m, m') describes one of the four quantum trefoils and $(t = 1/2, t_3, t_0)$ describes one of the four fermion families (leptons, neutrinos, down quarks, up quarks). Eq. (11.2)

records a one-to-one correspondence between the four quantum trefoils and the four fermion families.

The relation (11.2) is next tentatively extended to hold for all particles. Then for elementary fermions $j = 3t = 3/2$, and for the triplet of weak bosons $(w^+w^-w^0)$, one has $j = 3t = 3$, while for the fourth weak boson $j = 3t = 0$. The particles belonging to the fundamental ($j = 1/2$) and adjoint ($j = 1$) representations of $SLq(2)$ are new particles that may be called preons, with values of (t, t_3, t_0) given by (11.2).

The set (t, t_3, t_0) are defined as indices of $SLq(2)$ but $Q = (t_3+t_0)e$ and t_0 have their usual physical meaning as charge and hypercharge. One then finds that the particles with higher values of (t, t_3, t_0) may be regarded as built up of the four preons (one charged, one neutral, and their antiparticles) belonging to the fundamental representation of $SLq(2)$. These composite particles composed of preons are also characterized by (11.1), where $(N, w, \pm r + 1)$ may be interpreted to describe a quantized knotted field binding the preons together. Both the field and particle aspects of the composite particle express the $SLq(2)$ symmetry.

In this way the intuitive trefoil picture, when implemented empirically as the $j = 3/2$ representation of the knot algebra, leads naturally to the fundamental ($j = 1/2$) representation of $SLq(2)$ and the preonic constructions. This development resembles the transition from the “8-fold way”, the adjoint representation of $SU(3)$, to the fundamental representation of $SU(3)$ and the quark constructions.

An unsatisfactory feature of the model, however, is the meaning of q , which is obscure. Like Planck’s constant, which normalizes the non-Abelian Heisenberg algebra, the parameter q also normalizes a non-Abelian algebra, but an algebra dependent on ε_q instead of i where ε_q is a different square root of -1 . Unlike h , which has the dimensions of an action, q is dimensionless.

The $SLq(2)$ introduction of substructure for the fermionic fields in terms of preons carrying quanta of charge $(e/3)$ resembles the Planck-Einstein introduction of substructure for the Maxwell field in terms of photons carrying quanta of energy $h\nu$. This analogy suggests a comparison of the $SLq(2)$ algebra, determined by q , with the Heisenberg algebra, determined by h , and may be based on the following quadratic form¹⁰ invariant under $SLq(2)$

transformations:

$$K = A^t \varepsilon_q A \quad (11.3)$$

where

$$\varepsilon_q = \begin{pmatrix} 0 & q^{-1/2} \\ -q^{1/2} & 0 \end{pmatrix} \quad \varepsilon_q^2 = -1 \quad (11.4)$$

K is invariant under $SLq(2)$ transformations of A :

$$A' = T A \quad T \in SLq(2) \quad (11.5)$$

Choosing

$$A = \begin{pmatrix} D_x \\ x \end{pmatrix} \quad (11.6)$$

and normalizing

$$K = q^{-1/2} \quad (11.7)$$

one has by (11.3) the following $SLq(2)$ invariant relation

$$D_x x - q x D_x = 1 \quad (11.8)$$

Equation (11.8) is satisfied if D_x is chosen as the q -difference operator, namely

$$D_x \psi(x) = \frac{\psi(qx) - \psi(x)}{qx - x} \quad (11.9)$$

If we introduce

$$P_x = \frac{\hbar}{i} D_x \quad (11.10)$$

then (11.8) becomes

$$(P_x x - q x P_x) \psi(x) = \frac{\hbar}{i} \psi(x) \quad (11.11)$$

If $q \rightarrow 1$, then (11.11) becomes the Heisenberg commutator applied to a quantum state. If q is near unity (as it must be insofar as the standard theory ($q = 1$) is approximately correct)

then by (11.9) D_x resembles the differentiation operator on a lattice space and q may play the role of a dimensionless regulator.

In view of the physical evidence suggestive of substructure, which has been described here, as well as the natural appearance of the non-standard q - derivative, it may be possible to utilize $SLq(2)$ to describe a finer level of structure than is currently considered.

We have ignored the gravitational field in this paper since it is not immediately relevant. As we have, however, discussed the knot symmetries of the fundamental particles, we have thereby also discussed the knot symmetries of these sources of the gravitational field. Since one expects that the symmetries of its source would in some measure be inherited by the gravitational field itself, it is interesting that knot states have emerged in a natural way from and are therefore compatible with attempts to quantize general relativity.¹¹

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A POSSIBLE $SL_q(2)$ SUBSTRUCTURE OF THE STANDARD MODEL

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ABSTRACT: We examine a quantum group extension of the standard model with the symmetry $SU(3) \times SU(2) \times U(1) \times$ global $SL_q(2)$. The quantum fields of this extended model lie in the state space of the $SL_q(2)$ algebra. The normal modes or field quanta carry the factors $D_{mm'}^j(q|abcd)$, which are irreducible representations of $SL_q(2)$ (which is also the knot algebra). We describe these field quanta as quantum knots and set $(j, m, m') = 1/2(N, w, \pm r + 1)$ where the (N, w, r) are restricted to be (the number of crossings, the writhe, the rotation) respectively, of a corresponding classical knot.

There is an empirical one-to-one correspondence between the four quantum trefoils and the four families of elementary fermions, a correspondence that may be expressed as $(j, m, m') = 3(t, -t_3, -t_0)$, where the four quantum trefoils are labelled by (j, m, m') and where the four families are labelled in the standard model by the isotopic and hypercharge indices (t, t_3, t_0) . We propose extending this correlation to all representations by attaching $D_{-3t, -3t_0}^{3t}(q|abcd)$ to the field operator of every particle labelled by (t, t_3, t_0) in the standard model. Then the elementary fermions ($t = 1/2$) belong to the $j = 3/2$ representation of $SL_q(2)$. The elements of the fundamental representation $j = 1/2$ will be called preons and $D_{-3t, -3t_0}^{3t}$ may be interpreted as describing the creation operator of a composite particle composed of elementary preons. $D_{mm'}^j$ may also be interpreted to describe a quantum knot when expressed as $D_{\frac{w}{2}, \frac{\pm r + 1}{2}}^{N/2}$. These complementary descriptions may be understood as describing a composite particle of N preons bound by a knotted boson field with N crossings.

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1 Introduction

The general notion that the elementary particles are topologically stabilized has had a long history beginning with Kelvin.¹ In recent times a classical knot related to the Skyrme soliton has been described by Faddeev and Niemi.² There is also an apparently unrelated preon literature beginning with the work of Pati and Salam.³ It turns out, however, that the knot and preon conjectures are not unrelated but may be formulated as complementary field and particle expressions of $SLq(2)$ symmetry.

To review the knot conjecture first note that the familiar knots of magnetic fields are macroscopic manifestations of the electroweak field. It is then natural to consider knots of electroweak field that are microscopic and quantized as well. Since these would be observed as solitonic in virtue of both their topological and quantum stability, it is also natural to ask if the known elementary particles might also be quantized knots of field. If they are, one expects that the most elementary particles, namely the elementary fermions, are also the most elementary quantum knots, namely the quantum trefoils. This possibility is suggested by the fact that there are 4 quantum trefoils and 4 classes of elementary fermions, and is supported by a unique one-to-one correspondence between the topological characterization of the 4 quantum trefoils and the quantum numbers of the 4 fermionic classes. To define a quantum knot we shall first record the irreducible representations of the knot algebra ($SLq(2)$).

2 Irreducible Representations of the Knot Algebra^{4,13}

The $2j + 1$ dimensional representation of $SLq(2)$ may be written as follows:

$$D_{mm'}^j(a, b, c, d) = \sum_{\substack{0 \leq s \leq n_+ \\ 0 \leq t \leq n_-}} \mathcal{A}_{mm'}^j(q, s, t) \delta(s + t, n'_+) a^s b^{(n_+ - s)} c^t d^{(n_- - t)} \quad (2.1)$$

where

$$n_{\pm} = j \pm m \quad (2.2)$$

$$n'_{\pm} = j \pm m' \quad (2.3)$$

and the arguments (a, b, c, d) satisfy the knot algebra:⁵

$$\begin{aligned} ab = qba \quad bd = qdb \quad bc = cb \quad ad - qbc = 1 \\ ac = qca \quad cd = qdc \quad da - q_1cb = 1 \end{aligned} \quad (\text{A})$$

where $q_1 = q^{-1}$.

The $\mathcal{A}_{mm'}^j$ are q -deformations of the Wigner coefficients that appear in irreducible representations of $SU(2)$.

The knot algebra (A) and hence $D_{mm'}^j(a, b, c, d)$ are defined only up to the gauge transformation

$$\begin{aligned} U_a(1) : \quad a' = e^{i\varphi_a} a \\ d' = e^{-i\varphi_a} d \end{aligned} \quad U_b(1) : \quad b' = e^{i\varphi_b} b \\ c' = e^{-i\varphi_b} c \quad (2.4)$$

Eqns (2.4) leave the algebra (A) invariant and induce on the elements of every representation the following $U_a(1) \times U_b(1)$ gauge transformation⁶

$$D_{mm'}^j(a', b', c', d') = e^{i(\varphi_a + \varphi_b)m} e^{i(\varphi_a - \varphi_b)m'} D_{mm'}^j(a, b, c, d) \quad (2.5)$$

3 Quantization of the Knot

Following the example of the quantization of angular momentum by representations $D_{mm'}^j$ of $SU(2)$ where the indices (j, m, m') refer to components of the angular momentum, we shall quantize the kinematics of the knot with representations of the knot algebra, also denoted by $D_{mm'}^j$, where the indices on $D_{mm'}^j$ are now related to (N, w, r) , the number of crossings, the writhe, and the rotation of the corresponding classical knot,⁷ by

$$\begin{aligned} j &= \frac{N}{2} \\ m &= \frac{w}{2} \\ m' &= \frac{\pm r + 1}{2} \end{aligned} \quad (3.1)$$

The relations (3.1) satisfy the following restrictions:

- (a) Of the set (N, w, r) only N is never negative and therefore corresponds to j which is also never negative.

(b) Half-integer representations require the factor $1/2$.

(c) $(2m)$ and $(2m')$, belonging to the same representation, are of the same parity, while the knot constraints require w and r to be of opposite parity.

Since the spectra of (j, m, m') are restricted by $SLq(2)$, and the spectra of (N, w, r) are restricted by knot topology, the states of the quantized knot are thus jointly restricted by both $SLq(2)$ and the knot topology. The equations (3.1) then establish a correspondence between a quantized knot described by $D_{\frac{w \pm r + 1}{2}}^{N/2}$ and a classical knot described by (N, w, r) , but the correspondence is not one-to-one.

Let

$$\Psi_{\frac{w \pm r + 1}{2}}^{N/2} = D_{\frac{w \pm r + 1}{2}}^{N/2} \sum_n c_n |n\rangle \quad (3.2)$$

where $\sum c_n |n\rangle$ lies in the state space defined by the knot algebra (A). The states, $|n\rangle$, forming a basis in this space are eigenstates of the commuting elements, b and c , with eigenvalues $\sim q^{n(6)}$.

Here $\Psi_{\frac{w \pm r + 1}{2}}^{N/2}$ is intended to describe a generic quantum knot for which the Hamiltonian has not been specified. In (3.2) $D_{\frac{w \pm r + 1}{2}}^{N/2}$ is a kinematic factor that resembles the spherical harmonic factor in $Y_m^\ell(\theta, \varphi)R(r)$, an eigenstate of a spherically symmetric Hamiltonian.

There are, as usual, important differences between the quantum construction and its classical image. In particular there are only two classical trefoils: $(w, r) = (3, 2)$ and $(-3, 2)$ while there are four quantum trefoils labelled by

$$(j, m, m') = \left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}\right), \left(\frac{3}{2}, -\frac{3}{2}, \frac{3}{2}\right), \left(\frac{3}{2}, \frac{3}{2}, -\frac{1}{2}\right), \left(\frac{3}{2}, -\frac{3}{2}, -\frac{1}{2}\right)$$

where $m' = -\frac{1}{2}$ corresponds to $m' = \frac{-r+1}{2}$ by (3.1) with $r = 2$.

The classical trefoils $(w, 2)$ and $(w, -2)$ are topologically not distinguishable. The corresponding quantum trefoils $(w, 2)$ and $(w, -2)$, when realized as elementary fermions, are distinguished by different values of the hypercharge, as we shall see. In the following, when (w, r) refers to a quantum trefoil, r may have either sign, and we shall write simply $m' = \frac{r+1}{2}$. Then there is a one-to-one correspondence between the quantum trefoil and the 2d-projection of the 3d-classical knot.

There are also quantum states for which $j < \frac{3}{2}$. By (3.1) these correspond to a classical image for which $N < 3$ and do not qualify as classical knots but may be described as twisted loops. We shall see that these $j < \frac{3}{2}$ states may be realized as preons.

4 Field Theory and Charges of Quantum Trefoils

One may construct a field theory of the quantum trefoils $D_{mm'}^{3/2}$ by attaching $D_{mm'}^{3/2}$ to a standard fermion field operator $\psi(x)$ as follows:

$$\Psi_{mm'}^{3/2} = \psi(x) D_{mm'}^{3/2} \quad (4.1)$$

By (2.5), the field operator $\Psi_{mm'}^{3/2}$ also transforms under the gauge transformations $U_a(1) \times U_b(1)$. If the attachment (4.1) is made consistently for both fermionic and bosonic fields one may construct a modified standard action that is invariant under $U_a \times U_b$, as is shown in the appendix and in more detail in Ref. 12. This invariance of the field action is a physical requirement since the relabelling of the algebra described by (2.4) cannot affect the physics. Then in view of this invariance there will be by Noether's theorem one conserved charge associated with $U_a(1)$ and a second conserved charge associated with $U_b(1)$. Then by (2.5) and (3.1) these charges may be defined by

$$Q(w) \equiv -k_w m = -k_w \frac{w}{2} \quad w = \pm 3 \quad (4.2)$$

$$Q(r) \equiv -k_r m' = -k_r \frac{r+1}{2} \quad r = \pm 2 \quad (4.3)$$

and may be referred to as the writhe and rotation charges. Here k_w and k_r are undetermined constants with the dimensions of an electric charge. In terms of $Q(w)$ and $Q(r)$, the $U_a(1) \times U_b(1)$ transformations on $\Psi_{mm'}^{3/2}$ become

$$\Psi_{mm'}^{3/2'} = e^{\frac{-i}{k_w} Q(w) \varphi(w)} e^{\frac{-i}{k_r} Q(r) \varphi(r)} \Psi_{mm'}^{3/2} \quad (4.4)$$

where $\varphi(w) = \varphi_a + \varphi_b$ and $\varphi(r) = \varphi_a - \varphi_b$ by (2.5). We next make a direct comparison between the $Q(w)$ and $Q(r)$ charges of the four quantum trefoils and the charge and hypercharge of the four fermion families of the standard theory each denoted by (f_1, f_2, f_3) in

Table 4.1,⁴ since we expect that the simplest elementary particles are quantum trefoils if a knot model is plausible. The knot entries in the table are determined by (3.1), (4.2), and (4.3).

Table 4.1

<u>Standard Representation</u>					<u>Trefoil Representation</u>				
(f_1, f_2, f_3)	\underline{t}	\underline{t}_3	\underline{t}_0	\underline{Q}_e	(w, r)	$\underline{D}_{\frac{w}{2} \frac{r+1}{2}}^{N/2}$	\underline{Q}_w	\underline{Q}_r	$\underline{Q}_w + \underline{Q}_r$
$(e, \mu, \tau)_L$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-e$	(3,2)	$D_{\frac{3}{2} \frac{3}{2}}^{3/2}$	$-k \left(\frac{3}{2}\right)$	$-k \left(\frac{3}{2}\right)$	$-3k$
$(\nu_e, \nu_\mu, \nu_\tau)_L$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	(-3,2)	$D_{-\frac{3}{2} \frac{3}{2}}^{3/2}$	$-k \left(-\frac{3}{2}\right)$	$-k \left(\frac{3}{2}\right)$	0
$(d, s, b)_L$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{3}e$	(3,-2)	$D_{\frac{3}{2} -\frac{1}{2}}^{3/2}$	$-k \left(\frac{3}{2}\right)$	$-k \left(-\frac{1}{2}\right)$	$-k$
$(u, c, t)_L$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{2}{3}e$	(-3,-2)	$D_{-\frac{3}{2} -\frac{1}{2}}^{3/2}$	$-k \left(-\frac{3}{2}\right)$	$-k \left(-\frac{1}{2}\right)$	$2k$

In Table (4.1) we have assumed a single value of k :

$$k_r = k_w = k \quad (4.5)$$

which is also the same for all trefoils. If we set $k = e/3$, we find that the four fermion families are related to the four quantum trefoils as follows:

$$Q_w = et_3 \quad (4.6)$$

$$Q_r = et_0 \quad (4.7)$$

$$Q_w + Q_r = Q_e \quad (4.8)$$

in agreement with the standard model where there is the independent relation for the electric charge

$$Q_e = e(t_3 + t_0) \quad (4.9)$$

If one aligns the trefoils and the fermion families in any order different from that in Table 4.1, one needs more than a single value of k . It is important that we choose $k_r = k_w$ and that we choose a single value of k for the four quantum trefoils. Note that it is also not

possible to exchange t_3 and t_0 in (4.6) and (4.7). Therefore the correspondence between the four fermion families and the four trefoils is empirically fixed and unique. This complete correspondence justifies the representation of each of the four fermion families by a quantum trefoil.

Since this correspondence is the basis of the following development, it is worth emphasizing that there is only one assumption on which the interpretation of the Table 4.1 is based: namely, that there is a single value of k for both Q_w and Q_r , and for all the four families. (There are in fact no grounds for choosing more than one value of k .) If we therefore postulate a unique value of k , that is enough to determine a unique correspondence between the four families and the four quantum trefoils, since only for this unique correspondence is there strict proportionality between (t_3, t_0, Q_e) and $(Q_w, Q_r, Q_w + Q_r)$ respectively. The value of k as $\frac{e}{3}$ then follows from the identification of the total charge of the trefoil, $Q_w + Q_r$, with Q_e , the electric charge of the fermion. Then $j_\mu^e = j_\mu^w + j_\mu^r$.

The correspondence between the quantum trefoils and the elementary fermions may be summarized by the following relations which may also be read directly from Table (4.1).

$$t = \frac{N}{6} \quad t = \frac{j}{3} \quad (4.10)$$

$$t_3 = -\frac{w}{6} \quad t_3 = -\frac{m}{3} \quad (4.11)$$

$$t_0 = -\frac{r+1}{6} \quad t_0 = -\frac{m'}{3} \quad (4.12)$$

$$Q_e = -\frac{e}{6}(w+r+1) \quad Q_e = -\frac{e}{3}(m+m') \quad (4.13)$$

Note also that

$$Q_e = -\frac{e}{N} \left(\frac{w+r+1}{2} \right) \quad Q_e = -e \frac{(m+m')}{2j} \quad (4.14)$$

holds for all the elementary fermions.

While Q, t_3 and t_0 are defined in the standard model with respect to $SU(2) \times U(1)$, here Q, t_3 and t_0 are defined with respect to the gauge transformations $U_a(1) \times U_b(1)$ of the knot algebra. They are also described by (w, r) and (m, m') as shown in Table (4.1) and Eqs. (4.11)-(4.13).

The kinematic factors $D_{m,m'}^j$ labelled by quantum numbers (j, m, m') or (t, t_3, t_0) are multinomials lying in the knot algebra (A) and are explicitly given by (2.1). These multinomials are associated with the knot (N, w, r) and, like the Jones polynomial, label the knot.

We incorporate Eqs. (4.10)-(4.12) into (4.1) as follows

$$\Psi^{1/2}(t_3, t_0, n) = \psi^{1/2}(t_3, t_0, n) D_{-3t_3-3t_0}^{3/2} |n\rangle \quad (4.15)$$

where $\psi^{1/2}(t_3, t_0, n)$ is the quantum field of the standard model that represents the fermion with electroweak $SU(2) \times U(1)$ quantum numbers $t = 1/2$ and (t_3, t_0) . Here $|n\rangle$ lies in the state space defined by the knot algebra where $n = 0, 1, 2$ labels the generation, e.g. (e, μ, τ) . Then $D_{-3t_3-3t_0}^{3/2} |n\rangle$ may be regarded as an “internal state function” reminiscent of a classical knot and providing substructure to the elementary fermion fields of the standard model.

We shall now propose that the non-trivial correspondence embodied in Table (4.1) and expressed by (4.15) for the elementary fermions holds more generally in the following form

$$\Psi_{t_3 t_0}^t(n) = \psi(t, t_3, t_0, n) D_{-3t_3-3t_0}^{3t} |n\rangle \quad (4.16)$$

i.e., we assume that (t, t_3, t_0) are related to (j, m, m') just as in the special case $t = \frac{1}{2}$:

$$3t = j \quad (4.17j)$$

$$3t_3 = -m \quad (4.17m)$$

$$3t_0 = -m' \quad (4.17m')$$

In other words we assume that there is an underlying $SL_q(2)$ symmetry of the elementary particles that may be expressed through the internal state functions $D_{mm'}^j |n\rangle$. For $j \geq 1$ not all states (m, m') of $D_{mm'}^j$ are filled. The occupied states are labelled by $D_{-3t_3-3t_0}^{3t}$ according to (4.17) and are determined by the intersection of the electroweak $SU(2) \times U(1)$ and the $SU_q(2)$ symmetries. The $|n\rangle$ are intended to represent the possible states of excitation of the quantum knot. The analogue of Table 4.1 for the elementary fermions is Table 4.2 for the elementary bosons of the Weinberg-Salam model.

Table 4.2

	\underline{t}	$\underline{t_3}$	$\underline{t_0}$	$\underline{D_{-3t_3-3t_0}^{3t}}$
W^+	1	1	0	D_{-30}^3
W^-	1	-1	0	D_{30}^3
W^3	1	0	0	D_{00}^3
W^0	0	0	0	D_{00}^0

We adopt the following rule:

If a particle is labelled in the standard model by electroweak quantum numbers (t, t_3, t_0) then attach to the quantum field operator of that particle the factor $D_{-3t_3-3t_0}^{3t}(a, b, c, d)$. This factor is to be understood as an element of the $j = 3t$ representation of the $SL_q(2)$ algebra and may be interpreted as the replacement of the point particle of the standard model by a solitonic structure described solely by this factor. The extension of (4.15) to (4.16) expresses the conservation of t_3 and t_0 everywhere in the modified standard model as a joint consequence of the $U_a \times U_b$ and $SU(2) \times U(1)$ invariance.

5 The Electroweak Interactions⁸

In the $SL_q(2)$ model the solitonic fermions interact by the emission and absorption of solitonic bosons. Denote the generic fermion-boson interaction by

$$\bar{\mathcal{F}}'' \mathcal{B}' \mathcal{F} \quad (5.1)$$

where

$$\mathcal{F} = \mathbf{F}(p, s, t_3, t_0) \left(D_{-3t_3-3t_0}^{3/2} \right) |n > \quad (5.2)$$

$$\bar{\mathcal{F}}'' = < n'' | \left(\bar{D}_{-3t_3-3t_0}^{3/2} \right)'' \bar{\mathbf{F}}''(p, s, t_3, t_0) \quad (5.3)$$

$$\mathcal{B}' = \mathbf{B}'(p, s, t_3, t_0) \left(D_{-3t_3-3t_0}^3 \right)' \quad (5.4)$$

and the pair (p, s) refer to momentum and spin. Then (5.1) becomes

$$(\bar{\mathbf{F}}'' \mathbf{B}' \mathbf{F}) < n'' | \bar{D}_{-3t_3''-3t_0''}^{3/2} D_{-3t_3'-3t_0'}^3 D_{-3t_3-3t_0}^{3/2} | n > \quad (5.5)$$

The matrix elements of the standard model will then be modified by the following form factors.

$$< n'' | \bar{D}_{-3t_3''-3t_0''}^{3/2} D_{-3t_3'-3t_0'}^3 D_{-3t_3-3t_0}^{3/2} | n > \quad (5.6)$$

Here n and n'' take on the values 0, 1, 2 corresponding to the 3 generations in each family of fermions. These form factors are 2 parameter numerical functions that are in principle observable.⁸ To calculate them one needs the solitonic factors $D_{mm'}^j(a, b, c, d)$ shown in Tables (5.1) and (5.2).

Table 5.1

(f_1, f_2, f_3)	\underline{t}	$\underline{t_3}$	$\underline{t_0}$	\underline{Q}	$\underline{D_{-3t_3-3t_0}^{3t}}$
(e, μ, τ)	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-e$	$D_{\frac{3}{2}\frac{3}{2}}^{3/2} \sim a^3$
$(\nu_e, \nu_\mu, \nu_\tau)$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	$D_{-\frac{3}{2}\frac{3}{2}}^{3/2} \sim c^3$
(d, s, b)	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{3}e$	$D_{\frac{3}{2}-\frac{1}{2}}^{3/2} \sim ab^2$
(u, c, t)	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{2}{3}e$	$D_{-\frac{3}{2}-\frac{1}{2}}^{3/2} \sim cd^2$

Table 5.2

	\underline{t}	$\underline{t_3}$	$\underline{t_0}$	\underline{Q}	$\underline{D_{-3t_3-3t_0}^{3t}}$
W^+	1	1	0	e	$D_{-30}^3 \sim c^3 d^3$
W^-	1	-1	0	$-e$	$D_{30}^3 \sim a^3 b^3$
W^3	1	0	0	0	$D_{00}^3 \sim f_3(b, c)$
W^0	0	0	0	0	$D_{00}^0 \sim f_0(b, c)$

The solitonic factors are computed according to (2.1) and are all monomials except for the neutral W^0 and W^3 . The numerical factors $\mathcal{A}_{mm'}^3$ have been dropped but may be computed according to

$$\mathcal{A}_{mm'}^j = \left[\frac{\langle n'_+ \rangle_{q_1}! \langle n'_- \rangle_{q_1}!}{\langle n_+ \rangle_{q_1}! \langle n_- \rangle_{q_1}!} \right]^{\frac{1}{2}} \begin{matrix} \langle n_+ \rangle \\ s \end{matrix}_{q_1} \begin{matrix} \langle n_- \rangle \\ t \end{matrix}_{q_1} \quad (5.7)$$

where

$$\begin{matrix} \langle n \rangle \\ s \end{matrix}_q = \frac{\langle n \rangle_q!}{\langle n-s \rangle_q! \langle s \rangle_q!} \quad \text{with} \quad \langle n \rangle_q = \frac{q^n - 1}{q - 1}; \quad q_1 = q^{-1} \quad (5.8)$$

Since we require that the fermion-boson interaction be expressed by (5.1), and that the total action be invariant under both $SU(2) \times U(1)$ and $U_a(1) \times U_b(1)$, (5.1) and (5.6) must share this invariance. Then since (4.17*m*) and (4.17*m'*) hold for the elementary \mathcal{F} , the same equations must also hold for \mathcal{B} , as noted in the Appendix. Hence Eq. (4.17*m*) and (4.17*m'*) are not simply conjectured extensions but they are an essential requirement of the model. We are also imposing (4.17*j*) but this is not required by the same argument.

Although $(N, w, r+1) = 6(t_1, -t_3, -t_0)$ is satisfied for the elementary fermions, it does not hold in general. For example, there is a ditrefoil realization of this relation for the weak bosons W^\pm but not for the neutral W^3 and W^0 . There is, however, always a $SL_q(2)$ realization by $(j, m, m') = 3(t, -t_3, -t_0)$ and the connection to the standard model is through (t, t_3, t_0) .

6 The Preon Representations.⁴

The elementary fermions already discussed are found in the $j = 3/2$ representation while the electroweak bosons lie in the $j = 3$ representation. We shall now consider the adjoint ($j = 1$) and fundamental ($j = 1/2$) representations. These are shown in Tables (6.1) and (6.2) again calculated with (2.1) but ignoring the numerical coefficients.

Table 6.1

$D^{1/2} :$	$m \backslash m'$	$\frac{1}{2}$	$-\frac{1}{2}$
	$\frac{1}{2}$	a	b
	$-\frac{1}{2}$	c	d

Table 6.2

$D^1 :$	$m \backslash m'$	1	0	-1
	1	a^2	ab	b^2
	0	ac	$ad + bc$	bd
	-1	c^2	cd	d^2

We shall refer to the members of the $D^{1/2}$ and D^1 representations as preons and bosonic preons respectively.

To determine (t_3, t_0, Q) for the preons and bosonic preons we shall extend the relations (4.17) empirically established for the elementary fermions, then extended to the electroweak bosons and generally embodied in $D_{-3t_3-3t_0}^{3t}$. The results for preons and bosonic preons are shown in Tables (6.3) and (6.4).

Table 6.3

Fermionic Preons $t = 1/6$

	t	t_3	t_0	Q
a	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{e}{3}$
b	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	0
d	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{e}{3}$
c	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	0

Table 6.4

Bosonic Preons $t = 1/3$

	$\underline{t_3}$	$\underline{t_0}$	$\underline{Q/e}$	$\underline{D_{mm'}^1}$		$\underline{t_3}$	$\underline{t_0}$	$\underline{Q/e}$	$\underline{D_{mm'}^1}$		$\underline{t_3}$	$\underline{t_0}$	$\underline{Q/e}$	$\underline{D_{mm'}^1}$
D_{11}^1	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{2}{3}$	a^2	D_{01}^1	0	$-\frac{1}{3}$	$-\frac{1}{3}$	ac	D_{-11}^1	$\frac{1}{3}$	$-\frac{1}{3}$	0	c^2
D_{10}^1	$-\frac{1}{3}$	0	$-\frac{1}{3}$	ab	D_{00}^1	0	0	0	$ad + bc$	D_{-10}^1	$\frac{1}{3}$	0	$\frac{1}{3}$	cd
D_{1-1}^1	$-\frac{1}{3}$	$\frac{1}{3}$	0	b^2	D_{0-1}^1	0	$\frac{1}{3}$	$\frac{1}{3}$	bd	D_{-1-1}^1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	d^2

By Table (6.3) there are two preons, a and b , charged and neutral, respectively, and their respective antiparticles, d and c . Other particles may be regarded as built out of the preons (a, b, c, d) since the values of (t_3, t_0, Q) of all these composite particles may be obtained by adding the (t_3, t_0, Q) of each of the constituent preons. Therefore the factors $D_{-3t_3-3t_0}^{3t}$ may be read in two ways: (a) as describing creation operators for quantum knots representing the internal state of a composite particle or (b) as a product of creation operators for the component preons.

The preceding remarks are illustrated in Tables (5.1), (5.2), (6.3) and (6.4) where $D_{-3t_3-3t_0}^{3t}$ and (Q, t, t_3, t_0) are summarized for preons, bosonic preons, elementary fermions, and weak bosons.

To show that the preon interpretation holds in all representations let us rewrite (2.1) by introducing (n_a, n_b, n_c, n_d) the exponents of (a, b, c, d) :

$$n_a = s \tag{6.1}$$

$$n_b = n_+ - s$$

$$n_c = t \tag{6.2}$$

$$n_d = n_- - t$$

Then

$$n_+ = n_a + n_b \tag{6.3}$$

$$n_- = n_c + n_d$$

$$n'_+ = n_a + n_c \tag{6.4}$$

$$n'_- = n_b + n_d$$

and

$$n_a + n_b + n_c + n_d = n_+ + n_- = 2j (= N = 6t) \tag{6.5}$$

$$n_a + n_b - n_c - n_d = n_+ - n_- = 2m (= w = -6t_3) \tag{6.6}$$

$$n_a + n_c - n_b - n_d = n'_+ - n'_- = 2m' (= r + 1 = -6t_0) \tag{6.7}$$

In the preon interpretation of $D_{mm'}^j$ the (a, b, c, d) are regarded as creation operators for (a, b, c, d) particles and the (n_a, n_b, n_c, n_d) are the numbers of (a, b, c, d) preons in each term.

These will vary from term to term but the left sides of Eqs. (6.5)-(6.7) remain the same in all terms contributing to $D_{mm'}^j$ and they also have simple meanings.

Eqs. (6.5)-(6.7) may be rewritten as

$$t = \frac{1}{6}(n_a + n_b + n_c + n_d) \quad (6.8)$$

$$t_3 = -\frac{1}{6}(n_a + n_b - n_c - n_d) \quad (6.9)$$

$$t_0 = -\frac{1}{6}(n_a - n_b + n_c - n_d) \quad (6.10)$$

By (4.9), (6.9) and (6.10)

$$Q = -\frac{e}{3}(n_a - n_d)$$

The representations already considered, (Tables (5.1), (5.2), and (6.3)) illustrate special cases of these general relations that express (t, t_3, t_0) of a composite particle in terms of the charges and hypercharges of the preonic constituents. In general when D_{mm}^j is not a monomial, the composite particle represented by $D_{-3t_0-3t_0}^{3t}$ is a superposition of distinct structures, all having the same (t, t_3, t_0) but with varying numbers (n_a, n_b, n_c, n_d) of preons.

7 The Complementary Models.⁹

The Eqs. (6.5), (6.6), (6.7), may also be read as knot relations

$$n_a + n_b + n_c + n_d = N \quad (7.1)$$

$$n_a + n_b - n_c - n_d = w \quad (7.2)$$

$$n_a + n_c - n_b - n_d = r + 1 \quad (7.3)$$

Eq. (7.1) states that the total number of preons equals the number of crossings (N). Since we shall assume that the preons are fermions, the knot is a fermion or boson depending on whether the number of crossings is odd or even.

The meaning of (7.2) and (7.3) becomes clearer if we note that a and d are antiparticles since they have opposite charge and hypercharge, while b and c are neutral antiparticles with opposite values of the hypercharge. We may therefore introduce the "preon numbers":

$$\nu_a = n_a - n_d \quad (7.4)$$

$$\nu_b = n_b - n_c \quad (7.5)$$

Then (7.2) and (7.3) may be rewritten as

$$\nu_a + \nu_b = w \quad (= -6t_3) \quad (7.6)$$

$$\nu_a - \nu_b = r + 1 \quad (= -6t_0) \quad (7.7)$$

By (7.6) and (7.7) the conservation of writhe and rotation, or the conservation of charge and hypercharge, is equivalent to the conservation of the preon numbers.

These considerations have led us to the position that the symmetry of a solitonic elementary particle, that is described by representations of the $SL_q(2)$ algebra, may be expressed in any of the following ways:

$$D_{mm'}^j = D_{-3t_3-3t_0}^{3t} = D_{\frac{w}{2}\frac{r+1}{2}}^{N/2} = \hat{D}_{\nu_a\nu_b}^{N'} \quad (7.8)$$

where N' is the total number of preons.

We interpret the different forms of $D_{mm'}^j$ as showing that different aspects of the solitonic particle all display the same $SL_q(2)$ symmetry.

In terms of (N, w, r)

$$D_{\frac{w}{2}\frac{r+1}{2}}^{N/2}(q|abcd) = \left[\frac{\langle n'_+ \rangle_{q_1}! \langle n'_- \rangle_{q_1}!}{\langle n_+ \rangle_{q_1}! \langle n_- \rangle_{q_1}!} \right]^{1/2} \sum_{\substack{0 \leq s \leq n_+ \\ 0 \leq t \leq n_-}} \left\langle \begin{matrix} n_+ \\ s \end{matrix} \right\rangle_{q_1} \left\langle \begin{matrix} n_- \\ t \end{matrix} \right\rangle_{q_1} \delta(s+t, n'_+) a^s b^{n_+-s} c^t d^{n_--t} \quad (7.9)$$

where

$$n_{\pm} = \frac{1}{2}[N \pm w] \quad (7.10)$$

$$n'_\pm = \frac{1}{2}[N \pm (r + 1)] \quad (7.11)$$

The complementary description expressed in terms of the population numbers (n_a, n_b, n_c, n_d) is

$$D_{mm'}^j = \hat{D}_{\nu_a \nu_b}^{N'} \quad (7.12)$$

where

$$\hat{D}_{\nu_a \nu_b}^{N'} = \left[\frac{\langle n_a + n_c \rangle_{q_1}! \langle n_b + n_d \rangle_{q_1}!}{\langle n_a + n_b \rangle_{q_1}! \langle n_c + n_d \rangle_{q_1}!} \right]^{1/2} \sum_{\substack{N' \geq n_a, n_b \geq 0 \\ N' \geq n_c, n_d \geq 0}} \left\langle \begin{matrix} n_a + n_b \\ n_a \end{matrix} \right\rangle_{q_1} \left\langle \begin{matrix} n_c + n_d \\ n_c \end{matrix} \right\rangle_{q_1} a^{n_a} b^{n_b} c^{n_c} d^{n_d} \quad (7.13)$$

These complementary representations (7.9) and (7.13) are related by

$$\hat{D}_{\nu_a \nu_b}^{N'} = \sum_{Nwr} \delta(N', N) \delta(\nu_a + \nu_b, w) \delta(\nu_a - \nu_b, r + 1) D_{\frac{w}{2} \frac{r+1}{2}}^{N/2} \quad (7.14)$$

where N' is the number of preons and N is the number of crossings.

For the fundamental and adjoint representations we have $j = 1/2$ and $j = 1$ respectively and therefore $N = 1$ or $N = 2$, where N is the number of crossings. These do not describe knots, but they do describe twisted loops. We may still compute w and r in the same way as for knots. Although these twisted loops would not have the topological stability of knots, they could be prevented from unrolling by a dynamical stability of w and r or equivalently by the conservation of the preon numbers.

Viewed as a knot, a fermion becomes a boson when the number of crossings is changed by adding or subtracting a curl. This picture is consistent with the complementary view of a curl as an opened preon loop.

8 Gluon Charge.⁹

The previous considerations are based on electroweak physics. To describe the strong interactions it is necessary according to standard theory to introduce $SU(3)$ charge. We shall therefore assume that each of the four preon operators appears in triplicate (a_i, b_i, c_i, d_i) where $i = R, Y, G$, without changing the algebra (A) . These colored preon operators provide a basis for the fundamental representation of $SU(3)$ just as the colored quark operators

do in standard theory. To adapt the electroweak operators to the requirements of gluon fields we make the following replacements:

$$\text{leptons :} \quad a^3 \rightarrow \epsilon^{ijk} a_i a_j a_k \quad (8.1)$$

$$\text{neutrinos :} \quad c^3 \rightarrow \epsilon^{ijk} c_i c_j c_k \quad (8.2)$$

$$\text{down quarks :} \quad a_i g^{k\ell} b_k b_\ell \quad (8.3)$$

$$\text{up quarks :} \quad c_i g^{k\ell} d_k d_\ell \quad (8.4)$$

where g^{jk} is the group metric of $SU(3)$. Here $(i, j, k) = (R, Y, G)$ and $(a_i b_i c_i d_i)$ are creation operators for colored preons. Then the leptons and neutrinos are color singlets while the quark states correspond to the fundamental representation of $SU(3)$, as required by standard theory.

9 The Elementary Fermions as Preonic Trefoils⁹

Since the number of crossings equals the number of preons, one may speculate that there is one preon at each crossing if both preons and crossings are considered pointlike. If the pointlike crossings are labelled $(\vec{x}_1 \vec{x}_2 \vec{x}_3)$, then by (8.1)-(8.4) the wave functions of the trefoils representing leptons (ℓ), neutrinos (ν), down quarks (d), up quarks (u) are as follows:

$$\Psi_\ell(\vec{x}_1 \vec{x}_2 \vec{x}_3) = \epsilon^{ijk} \psi_i(a|\vec{x}_1) \psi_j(a|\vec{x}_2) \psi_k(a|\vec{x}_3) \quad (9.1)$$

$$\Psi_\nu(\vec{x}_1 \vec{x}_2 \vec{x}_3) = \epsilon^{ijk} \psi_i(c|\vec{x}_1) \psi_j(c|\vec{x}_2) \psi_k(c|\vec{x}_3) \quad (9.2)$$

$$\Psi_d(\vec{x}_1 \vec{x}_2 \vec{x}_3) = \psi_i(a|\vec{x}_1) g^{jk} \psi_j(b|\vec{x}_2) \psi_k(b|\vec{x}_3) \quad (9.3)$$

$$\Psi_u(\vec{x}_1 \vec{x}_2 \vec{x}_3) = \psi_i(c|\vec{x}_1) g^{jk} \psi_j(d|\vec{x}_2) \psi_k(d|\vec{x}_3) \quad (9.4)$$

where $i = (R, Y, G)$ and $\psi_i(a|\vec{x}) \dots \psi_i(d|\vec{x})$ are colored δ -like functions localizing the preons at the crossings.

Then the wave function of a lepton describes a *singlet* trefoil particle containing three preons of charge $(-e/3)$ and hypercharge $(-e/6)$. The corresponding characterization of a neutrino describes a *singlet* trefoil containing three neutral preons of hypercharge $(-e/6)$.

The wave function of a down quark describes a colored trefoil particle containing one a preon with charge $(-e/3)$ and hypercharge $(-e/6)$ and two neutral b preons with hypercharge $(e/6)$. The corresponding characterization of an up-quark describes a colored trefoil containing two charged d preons with charges $(e/3)$ and hypercharge $(e/6)$, and one neutral c preon with hypercharge $(-e/6)$.

This hypothetical structure is held together by the trefoil of fields connecting the preons. A search for this kind of substructure depends critically on the mass of the conjectured preons and the strength with which they are bound.

10 Preons as Physical Particles.⁴

We have so far viewed the preons mainly as a simple way to describe the algebraic structure of the knot polynomials. If these preons are in fact physical particles, the following decay modes of the quarks are possible.

$$\text{Down quarks: } \mathcal{D}_{\frac{3}{2}-\frac{1}{2}}^{3/2} \rightarrow \mathcal{D}_{\frac{1}{2}\frac{1}{2}}^{1/2} + \mathcal{D}_{1-1}^1, \quad (ab^2 \rightarrow a + b^2)$$

or

$$\text{Up quarks: } \mathcal{D}_{-\frac{3}{2}-\frac{1}{2}}^{3/2} \rightarrow \mathcal{D}_{-\frac{1}{2}\frac{1}{2}}^{1/2} + \mathcal{D}_{-1-1}^1, \quad (cd^2 \rightarrow c + d^2)$$

and the preons could play an intermediary role as virtual particles in quark processes.

The simple knot model predicts an unlimited number of excited states⁸ but it appears that there are only three generations, e.g. (d, s, b) . According to the preon scenario, however, it may be possible to avoid this problem by showing that the quarks will dissociate into preons if given a critical “dissociation energy” less than that needed to reach the level of the fourth predicted flavor. In that case one would also expect the formation of a preon-quark plasma at sufficiently high temperatures.

It may be possible to study the thermodynamics of the plasma composed of quarks and these hypothetical particles.

Since the a and d particles are charged $(\pm e/3)$ one should expect their electro-production according to

$$e^+ + e^- \rightarrow a + \bar{a} \dots$$

at sufficiently high energies of a colliding (e^+, e^-) pair.

If the preons are assumed to be pointlike they must also be very heavy. If the trefoil model is considered seriously for the leptons and neutrinos, then the binding energy must nearly compensate the mass of the very heavy constituent preons with a correspondingly higher melting temperature for the leptons and neutrinos.

11 Summary and Comments

In this paper the quantum knot has been characterized kinematically by $D_{mm'}^j(abcd)$, an element of an irreducible representation of the knot algebra $SL_q(2)$ with

$$(j, m, m') = \frac{1}{2} (N, w, \pm r + 1) \quad (11.1)$$

where the spectrum of (j, m, m') is limited by $SL_q(2)$ and the spectrum of (N, w, r) is restricted by the topology of a classical knot. The pair (w, r) are topological constants of the classical motion and the pair (m, m') are quantum constants of the motion by virtue of the $U_a(1) \times U_b(1)$ invariance of the $SL_q(2)$ algebra. If an elementary particle is identified as a quantum knot, it acquires in addition to the familiar angular momentum and isotopic spin, new degrees of freedom associated with the knot algebra as represented by $D_{mm'}^j$.

When the 12 elementary fermions are described as 3 states of excitation of 4 quantum trefoils, each quantum trefoil corresponds to one family of 3 fermions. The correspondence is unique and is expressed by the empirical relation

$$(j, m, m') = 3(t, -t_3, -t_0) \quad (11.2)$$

where (j, m, m') describes one of the four quantum trefoils and $(t = 1/2, t_3, t_0)$ describes one of the four fermion families (leptons, neutrinos, down quarks, up quarks). Eq. (11.2) records a one-to-one correspondence between the four quantum trefoils and the four fermion families.

In order to conserve t_3 and t_0 in all interactions the relation (11.2) is next extended to hold for all particles. Then for elementary fermions $j = 3t = 3/2$, and for the triplet of weak bosons $(W^+W^-W^0)$, one has $j = 3t = 3$, while for the fourth weak boson $j = 3t = 0$.

The particles belonging to the fundamental ($j = 1/2$) and adjoint ($j = 1$) representations of $SL_q(2)$ are new particles that may be called preons, with values of (t, t_3, t_0) given by (11.2).

The set (t, t_3, t_0) are defined as indices of $SL_q(2)$ but $Q = (t_3 + t_0)e$ and t_0 have their usual physical meaning as charge and hypercharge. One then finds that the particles with higher values of (t, t_3, t_0) may be regarded as built up of the four preons (one charged, one neutral, and their antiparticles) belonging to the fundamental representation of $SL_q(2)$. These composite particles composed of preons are also characterized by (11.1), where $(N, w, \pm r + 1)$ may be interpreted to describe a quantized knotted field binding the preons together. Both the field and particle aspects of the composite particle express the $SL_q(2)$ symmetry.

In this way the intuitive trefoil picture, when implemented empirically as the $j = 3/2$ representation of the knot algebra, leads naturally to the fundamental ($j = 1/2$) representation of $SL_q(2)$ and the preonic constructions. This development resembles the transition from the “8-fold way”, the adjoint representation of $SU(3)$, to the fundamental representation of $SU(3)$ and the quark constructions.

An unsatisfactory feature of the model, however, is the meaning of q , which is obscure. Like Planck’s constant, which normalizes the non-Abelian Heisenberg algebra, the parameter q also normalizes a non-Abelian algebra, but an algebra dependent on ϵ_q instead of i where ϵ_q is a different square root of -1 . Unlike h , which has the dimensions of an action, q is dimensionless.

The introduction of substructure, determined by the $SL_q(2)$ algebra, for the quantum fields in terms of preons resembles the introduction of substructure for the quantum fields in terms of field quanta determined by the Heisenberg algebra holding for conjugate field operators. This analogy suggests a comparison of the $SL_q(2)$ algebra, determined by q , with the Heisenberg algebra, determined by h and may be based on the following quadratic form¹⁰ invariant under $SL_q(2)$ transformations:

$$K = A^t \varepsilon_q A \tag{11.3}$$

where

$$\varepsilon_q = \begin{pmatrix} 0 & q^{-1/2} \\ -q^{1/2} & 0 \end{pmatrix} \quad \varepsilon_q^2 = -1 \tag{11.4}$$

K is invariant under $SL_q(2)$ transformations of A :

$$A' = T A \quad T \in SL_q(2) \quad (11.5)$$

Choosing

$$A = \begin{pmatrix} D_x \\ x \end{pmatrix} \quad (11.6)$$

and normalizing

$$K = q^{-1/2} \quad (11.7)$$

one has by (11.3) the following $SL_q(2)$ invariant relation

$$D_x x - q x D_x = 1 \quad (11.8)$$

Equation (11.8) is satisfied if D_x is chosen as the q -difference operator, namely

$$D_x \psi(x) = \frac{\psi(qx) - \psi(x)}{qx - x} \quad (11.9)$$

If we introduce

$$P_x = \frac{\hbar}{i} D_x \quad (11.10)$$

then (11.8) becomes

$$(P_x x - q x P_x) \psi(x) = \frac{\hbar}{i} \psi(x) \quad (11.11)$$

If $q \rightarrow 1$, then (11.11) becomes the Heisenberg commutator applied to a quantum state. If q is near unity (as it must be insofar as the standard theory ($q = 1$) is approximately correct) then by (11.9) D_x resembles the differentiation operator on a lattice space and q may play the role of a dimensionless regulator.

In view of the physical evidence suggestive of substructure, which has been described here, as well as the natural appearance of the non-standard q -derivative, it may be possible to utilize $SL_q(2)$ to describe a finer level of structure than is currently considered.

We have ignored the gravitational field in this paper since it is not immediately relevant. As we have, however, discussed the knot symmetries of the fundamental particles, we have

thereby also discussed the knot symmetries of these sources of the gravitational field. Since one expects that the symmetries of its source would in some measure be inherited by the gravitational field itself, it is interesting that knot states have emerged in a natural way from and are therefore compatible with attempts to quantize general relativity.¹¹

12 Appendix¹²

Here we show the invariance of the modified action under the complete gauge group

$$\mathcal{S} = S \times s \tag{12.1}$$

where

$$\begin{aligned} S &= \text{local } SU(2) \times U(1) \\ s &= \text{global } U_a(1) \times U_b(1) \end{aligned}$$

The vector potential of the non-Abelian part of the standard model is

$$\begin{aligned} \mathcal{W}_\mu &= i g W_\mu^k t_k \quad k = (+, -, 3) \\ t_\pm &= \frac{1}{2}(\sigma_1 + i\sigma_2) \\ t_3 &= \sigma_3 \end{aligned} \tag{12.2}$$

The corresponding vector potential of the modified standard model may be chosen as

$$\mathcal{W}_\mu = i g W_\mu^k \tau_k \tag{12.3}$$

where, in the notation of Table (4.2),

$$\begin{aligned} \tau_k &= c_k t_k D_k \quad k = (+, -, 3) \\ D_+ &= D_{-30}^3 / \mathcal{A}_+ = \bar{b}^3 \bar{a}^3 \\ D_- &= D_{-30}^3 / \mathcal{A}_- = a^3 b^3 \\ D_3 &= D_{00}^3 = f(\bar{b}b) \end{aligned} \tag{12.4}$$

and where the c_k are free numerical constants. Here \mathcal{A}_+ and \mathcal{A}_- are the numerical factors appearing in (2.1). The covariant derivative and field strength are constructed in the familiar

way:

$$\nabla_\mu = \partial_\mu + \mathcal{W}_\mu \quad (12.5)$$

$$\mathcal{W}_{\mu\lambda} = [\nabla_\mu, \nabla_\lambda] \quad (12.6)$$

Then one may show that the modified field strength is¹²

$$\mathcal{W}_{\mu\lambda} = W_{\mu\lambda}^s \tau_s + \hat{W}_{\mu\lambda}^s D_s \quad (12.7)$$

where

$$W_{\mu\lambda}^s = ig(\partial_\mu W_\lambda^s - \partial_\lambda W_\mu^s) - g^2 f_{m\ell}^s(\bar{b}b) W_\mu^m W_\lambda^\ell \quad (12.8)$$

and

$$\hat{W}_{\mu\lambda}^s = -\frac{1}{2}g^2 \delta(\ell, \pm) \delta(m, \mp) \hat{f}_{m\ell}^s(\bar{b}b) W_\mu^m W_\lambda^\ell \quad (12.9)$$

The modified structure constants $f_{m\ell}^s(\bar{b}b)$ and $\hat{f}_{m\ell}^s(\bar{b}b)$ become numerical when evaluated on the ground state $|0\rangle$ of the q -oscillator.

The $\hat{f}_{m\ell}^s$ vanish unless $(m, \ell) = (\pm, \mp)$ and $s = 3$, and therefore $\hat{W}_{\mu\lambda}^s$ also vanishes unless $s = 3$.

We choose the modified field action to be

$$\langle 0 | \text{Tr } \mathcal{W}_{\mu\lambda} \mathcal{W}^{\mu\lambda} | 0 \rangle \quad (12.10)$$

where the trace is over the t_k matrices and $|0\rangle$ is the ground state of the q oscillator. The structure constants $f_{k\ell}^s(\bar{b}b)$ and $\hat{f}_{k\ell}^s(\bar{b}b)$ become numerical in (12.10).

After reducing the trace one finds

$$\text{Tr } \mathcal{W}_{\mu\lambda} \mathcal{W}^{\mu\lambda} = W_{\mu\lambda}^m W^{p\mu\lambda} \text{Tr } \tau_m \tau_p + 2 \hat{W}_{\mu\lambda}^m \hat{W}^{p\mu\lambda} D_m D_p \quad (12.11)$$

and

$$\mathcal{S}(\text{Tr } \mathcal{W}_{\mu\lambda} \mathcal{W}^{\mu\lambda}) \mathcal{S}^{-1} = c_m c_p W_{\mu\lambda}^m W^{p\mu\lambda} \text{Tr } t_m t_p (s D_m D_p s^{-1} + 2 \hat{W}_{\mu\lambda}^m \hat{W}^{p\mu\lambda} (s D_m D_p s^{-1})) \quad (12.12)$$

where $\mathcal{S} = S \times s$ and there is the standard invariance under S . The factors $s D_m D_p s^{-1}$ remain to be considered. In the first term on the right $\text{Tr } t_m t_p$ vanishes unless

$$(m, p) = (\pm, \mp) \quad \text{or} \quad (m, p) = (3, 3) \quad (12.13)$$

In both cases $D_m D_p$ is neutral and therefore $s D_m D_p s^{-1} = D_m D_p$. The second term on the right vanishes by (12.9) unless

$$(m, p) = (3, 3) \quad (12.14)$$

Again, since $D_m D_p$ is neutral, this term is also invariant. Therefore the modified self-interaction of the non-Abelian vector field remains invariant under $U_a \times U_b$ as well as under S .

To show the invariance of the modified Fermion-Boson interaction under $U_a \times U_b$, one also needs to consider the modification introduced by the following form factor

$$\bar{D}_{m''p''}^{j''} D_{mp}^j D_{m'p'}^{j'} \quad (12.15)$$

which multiplies the term that previously described the standard Fermion-Boson interaction. We shall impose $U_a(1) \times U_b(1)$ invariance on this form factor and hence on the modified interaction.

By (2.5) this invariance requires in (12.15):

$$(m, p) = (m'', p'') - (m', p') \quad (12.16)$$

But since the initial (m', p') and final states (m'', p'') represent fermions, the boson (m, p) is by (4.15)

$$(m, p) = -3[(t_3, t_0)'' - (t_3, t_0)'] \quad (12.17)$$

and since (t_3, t_0) are conserved in the standard model one has

$$(m, p) = -3(t_3, t_0) \quad (12.18)$$

for the intermediate boson as well as for the fermions. Eqs. (4.17m) and (4.17m') are therefore necessary conditions for $U_a(1) \times U_b(1)$ invariance in this model. Also in (12.15)

$$j' + j'' \geq j \geq |j' - j''|$$

and since $j \geq |m|$,

$$D_{\pm 30}^j = D_{\pm 30}^3$$

Therefore the invariance of (12.15) is not only required for the existence of the charges Q_a and Q_b but also implies

$$(j, m, m') = 3(t, -t_3, -t_0)$$

for the intermediate charged boson.

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